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Time optimal controllability for infinite dimensional systems

In the frame of the present project the following activities took place: documentation, updating the bibliography, conferences, scientic contacts, analysis, research and publication of the results. We have published a paper in an ISI journal and we have submitted for publication three other papers in ISI journals. Other two papers are in preparation.

The research objectives for the mentioned period of time are:

Stage 1. To establish new regularity results for the cost functions associated to a linear system.

Stage 2. To prove the equivalence between the minimum time problem and the minimum norm control associated to a linear system.

All these research objectives were realized. In the following we give a presentation of the main results obtained.

Let X and U be two Banach spaces and consider the control system represented by

$$y\left(t,x,u\right)=S\left(t\right)x+V\left(t\right)u,\ \ t>0 \tag{1}$$

$$y\left(0,x,u\right)=x,$$

where y is the state, t the time and u the control. Here, $\{S(t);\ t\geq 0\}$ is a C_0-semigroup on X and $\{V\left(t\right);\ t>0\}$ is a family of bounded linear operators, $V\left(t\right):L^{\infty}\left(0,t;U\right)\rightarrow X$, such that the following condition is satisfied

$$V\left(t_1+t_2\right)u=S\left(t_2\right)V\left(t_1\right)u+V\left(t_2\right)J_{t_1}u, \tag{2}$$

for all $t_1,t_2>0$ and $u\in L^{\infty}(0,t_1+t_2;U)$, where J_{t_1} is a translation operator given by

$$\left(J_{t_1}u\right)\left(s\right)=u\left(s+t_1\right) \tag{3}$$

for $s\geq 0$. Clearly, in $V\left(t_1\right)u$ we have considered the restriction of u to $\left[0,t_1\right]$. Further, assume that for each $u\in L^{\infty}(0,+\infty;U)$ we have that $t\mapsto V(t)u$ is continuous and $\lim_{t\to 0}V(t)u=0$.

The typical example is the distributed control system

$$y'(t)=Ay(t)+Bu(t), \tag{4}$$

where A is the generator of $\{S(t);\ t\geq 0\}$ and B is linear and bounded from U to X. In this case, $V(t)u=\int_0^t S(t-s)Bu(s)ds$. The operator B could be also unbounded to cover the boundary control systems.

For $r>0$ and $t>0$ define

$$\mathcal{U}_r\left(t\right)=\{u\in L^{\infty}(0,t;U);\ \ \|u\|_{\infty}\leq r\}.$$

Denote by $\mathcal{C}_r(t)$ the null controllable set at time $t > 0$, i.e., the set of all points $x \in X$ for which there exists $u \in \mathcal{U}_r(t)$ with $y(t,x,u) = 0$. Consider $\mathcal{C}_r(0) = \{0\}$ and set $\mathcal{C}_r = \bigcup_{t\geq 0} \mathcal{C}_r(t)$, called the null controllable set, and define the minimum time function $\mathcal{T} : (0,+\infty) \times X \to [0,+\infty]$ by

$$\mathcal{T}(r,x) = \begin{cases} \inf\{t \geq 0;\ x \in \mathcal{C}_r(t)\},\ \text{if } x \in \mathcal{C}_r \\ +\infty,\ \text{elsewhere.} \end{cases}$$

For $t > 0$ and $x \in X$, denote by $\mathcal{M}(t,x)$ the (possibly empty) set of controls $u \in L^\infty(0,t;U)$ such that $y(t,x,u) = 0$ and define the control cost to bring x to 0 as the function $\mathcal{E} : (0,+\infty) \times X \to [0,+\infty]$ given by

$$\mathcal{E}(t,x) = \inf_{u\in\mathcal{M}(t,x)} \|u\|_\infty .$$

The basic hypothesis we shall refer to in the sequel is the following.

(H) There exists $\gamma : (0,+\infty) \to (0,+\infty)$ such that

$$S(t)B(0,\gamma(t)) \subseteq V(t)B(0,1), \tag{5}$$

for any $t > 0$. Here, $B(0,\gamma(t))$ is the closed ball of radius $\gamma(t)$ from X, while $B(0,1)$ is the closed unit ball from $L^\infty(0,t;U)$, i.e., $\mathcal{U}_1(t)$.

By the open mapping theorem, (5) is equivalent to

$$\mathit{Range}\,(S(t)) \subseteq \mathit{Range}\,(V(t))\,,$$

which means that all points of X can be transferred to zero in time t by $L^\infty(0,t;U)$-controls.

We further state some additional hypotheses that we shall frequently use in the sequel.

(H1) X and U are reflexive Banach spaces.

(H2) For every $t > 0$, $V(t) = H(t)^*$ for some $H(t) : X^* \to L^1(0,t;U^*)$.

It is well known that for every C_0-semigroup $\{S(t);\ t \geq 0\}$ there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$, for any $t \geq 0$.

Propoziţia 0.0.1 *Assume **(H)**.*
(i) For any $r > 0$, $x \in X$ and $t > 0$, there exists $u^* \in \mathcal{U}_r(t)$ such that

$$\|y(t,x,u^*)\| \leq Me^{\omega t}\,|\|x\| - r\gamma(t)|\,. \tag{6}$$

(ii) If $\|x\| \leq r\gamma(t)$, then $x \in \mathcal{C}_r(t)$.
(iii) If $x_0 \in \mathcal{C}_r(t)$ and $\|x - x_0\| \leq \rho\gamma(t')$ for some $t' \in (0,t]$ and $\rho > 0$, then $x \in \mathcal{C}_{r+\rho}(t)$.
(iv) Let $0 < r_1 < r_2$. If $x \in \mathcal{C}_{r_1}(t)$, then there exists $s \in (0,t)$ such that $x \in \mathcal{C}_{r_2}(s)$.
(v) Let $0 < t_1 < t_2$. If $x \in \mathcal{C}_r(t_1)$, then there exists $\overline{r} \in (0,r)$ such that $x \in \mathcal{C}_{\overline{r}}(t_2)$.

In the following we give a result on the equivalence between the minimum time function and the minimum energy and on the monotonicity of these two functions. The proof follows using the estimates obtained in Proposition 0.0.1.

Teorema 1 *Suppose the existence of optimal controls for the minimum time and minimum energy problems. Assume **(H)**. Then,*
(i) if $0 < t_1 < t_2$ *and* $\mathcal{E}(t_1, x) > 0$*, then*

$$\mathcal{E}(t_2, x) < \mathcal{E}(t_1, x);$$

(ii) if $0 < r_1 < r_2$ *and* $x \in \mathcal{C}_{r_1}$*, then*

$$\mathcal{T}(r_2, x) < \mathcal{T}(r_1, x);$$

(iii) for any $r > 0$ *and* $x \in \mathcal{C}_r$ *we have*

$$\mathcal{E}(\mathcal{T}(r, x), x) = r; \tag{7}$$

(iv) for any $t > 0$ *and* $x \in X$ *we have*

$$\mathcal{T}(\mathcal{E}(t, x), x) = t.$$

Observaţia 0.0.1 *Theorem 1 says that, for* $r > 0$ *and* $x \in \mathcal{C}_r$*,* $\mathcal{T}(r, x)$ *is the unique solution of the equation* $\mathcal{E}(t, x) = r$*. Also, given* $t > 0$ *and* $x \in X$*,* $\mathcal{E}(t, x)$ *is the unique solution of* $\mathcal{T}(r, x) = t$*.*

From Theorem 1 we easily get the following result.

Corolarul 0.0.1 *Suppose the hypotheses of Theorem 1. Let* $t > 0$ *and* $u \in L^\infty(0, t; U)$ *a minimum norm control such that* $y(t, x, u) = 0$*. Then* u *is a time optimal control for* x*, under the norm constraint* $r = \mathcal{E}(t, x)$*. Conversely, let* $r > 0$ *and* $v \in \mathcal{U}_r(\mathcal{T}(r, x))$ *a time optimal control for* x*. Then* v *is a minimum norm control on* $[0, \mathcal{T}(r, x)]$*.*

Further, we give estimates of the minimum time function around points in the null controllable set and around points in the boundary of the null controllable set. Moreover, we get local uniform continuity of the minimum time function on the null controllable set.

In what follows we denote

$$M_\gamma := \sup_{s \in \mathbb{R}_+} \gamma(s) \in (0, +\infty].$$

Teorema 2 *Assume **(H)**. Let* $r > 0$*. Then, for any* $x \in X$ *with* $\|x\| < rM_\gamma$ *we have* $x \in \mathcal{C}_r$*. Assume further that the function* γ *in* (5) *is continuous, strictly increasing and* $\gamma(0) = 0$*.*
(i) For any for any $x \in X$ *with* $\|x\| < rM_\gamma$*,* $\mathcal{T}(r, x) \leq \gamma^{-1}(\|x\|/r)$*.*
(ii) Let $x \in \mathcal{C}_r$ *and* $z \in X$ *such that* $\|x - z\| < (r/M)\, e^{-\omega \mathcal{T}(r,x)} M_\gamma$*. Then* $z \in \mathcal{C}_r$ *and*

$$\mathcal{T}(r, z) \leq \mathcal{T}(r, x) + \gamma^{-1}\left(\frac{\|x - z\|}{r} M e^{\omega \mathcal{T}(r,x)}\right). \tag{8}$$

(iii) In the case $\omega > 0$*, if* $x \in \mathcal{C}_r$ *and* $z \notin \mathcal{C}_r$*, then*

$$\mathcal{T}(r, x) \geq -\frac{1}{\omega} \log\left(\frac{\|x - z\| M}{rM_\gamma}\right).$$

Consequently, $\lim_{x\to z}\mathcal{T}(r,x)=+\infty$, *for any* $z\in\partial\mathcal{C}_r$.
In the case $\omega\le 0$ *we have that* $X=\mathcal{C}_r$.
(iv) If $M_\gamma<+\infty$, *then* $\mathcal{C}_r$ *is open and for any* $x_0\in\mathcal{C}_r$ *we have*

$$|\mathcal{T}(r,z_1)-\mathcal{T}(r,z_2)|\le\gamma^{-1}\left(c_r\|z_1-z_2\|\right) \tag{9}$$

for any $z_1,z_2\in B(x_0,\delta_r)$, *where, in the case* $\omega>0$,

$$c_r=\frac{M}{r}e^{\omega\left(\mathcal{T}(r,x_0)+\gamma^{-1}(M_\gamma/k)\right)} \text{ and } \delta_r=\frac{rM_\gamma}{Mk}e^{-\omega(\mathcal{T}(r,x_0)+M_\gamma)} \tag{10}$$

for some $k>\max\{M_\gamma/\gamma(M_\gamma),2\}$ *and, in the case* $\omega\le 0$,

$$c_r=\frac{M}{r}\text{ and }\delta_r=\frac{rM_\gamma}{2M}. \tag{11}$$

(v) If $M_\gamma=+\infty$, *then* $\mathcal{C}_r=X$ *and, in the case* $\omega>0$, *for any* $x_0\in X$ *and any* $\delta>0$ *there exists*

$$c_r=\frac{M}{r}e^{\omega\left(\mathcal{T}(r,x_0)+\gamma^{-1}(M\delta/re^{\omega\mathcal{T}(r,x_0)})\right)} \tag{12}$$

such that (9) *holds for any* $z_1,z_2\in B(x_0,\delta)$. *In the case* $\omega\le 0$, (9) *holds for any* $z_1,z_2\in X$ *where* $c_r=M/r$.

One of the main results obtained is the continuity of the minimum time function $\mathcal{T}$, as a function of both variables. To prove that, we need to show first that $\mathcal{T}$ is continuous in the first variable.

Propoziţia 0.0.2 *Assume* ***(H)****. Moreover, assume that* U *is a reflexive Banach space and* ***(H2′)*** *holds. Let* $r_0>0$ *and* $x\in\mathcal{C}_{r_0}$. *Then* $\mathcal{T}(\cdot,x)$ *is continuous in* r_0.

Observaţia 0.0.2 *Since* $\mathcal{E}(\cdot,x)$ *and* $\mathcal{T}(\cdot,x)$ *are inverse one to another, under the hypotheses of Proposition 0.0.2 , we get that* $\mathcal{E}(\cdot,x)$ *is continuous on* $(0,+\infty)$. *Let us consider the function*

$$\mathcal{E}(t)=\sup_{\|x\|\le 1}\mathcal{E}(t,x),\ t>0,$$

which is lower semicontinuous. Then, we can get a function γ^* *satisfying* (5) *which is upper semicontinuous. Indeed, define*

$$\gamma^*(t)=\sup\{\gamma(t);\ \gamma \text{ satisfies } (5)\}.$$

It is easy to prove that

$$\mathcal{E}(t)=\frac{1}{\gamma^*(t)},$$

hence γ^* *is upper semicontinuous.*

Now, using Theorem 2 and Proposition 0.0.2, we can prove the main result of this section, on the continuity of $\mathcal{T}$.

Teorema 3 *Assume the hypotheses of Theorem* 2 *and* ***(H1)****. Let* $r_0 > 0$ *and* $x_0 \in X$ *be such that* $x_0 \in \mathcal{C}_{r_0}$*. Then the minimum time function* $\mathcal{T}$ *is continuous in* (r_0, x_0)*.*

We prove the Lipschitz continuity of the minimum energy function in the state variable.

Propoziţia 0.0.3 *Assume* ***(H)****. Then, for every* $x, z \in X$ *and every* $t > 0$*, we have*

$$|\mathcal{E}(t,x) - \mathcal{E}(t,z)| \leq \frac{1}{\gamma(t)} \|x - z\|.$$

Moreover, we studied two main situations when the limit of Pareto minima of a sequence of perturbations of a set-valued map F is a critical point of F. The concept of criticality is understood in the Fermat generalized sense by means of limiting (Mordukhovich) coderivative. Firstly, we consider perturbations of enlargement type which, in particular, cover the case of perturbation with dilating cones. Secondly, we study the case of Aubin type perturbations, and for this we introduce and study a new concept of openness with respect to a cone.

We briefly present next the main results.

Let $K \subset Y$ be a convex closed cone and, additionally, we suppose it is as well pointed (that is, $K \cap -K = \{0\}$) and proper (that is, $K \neq \{0\}$). Consider $F : X \rightrightarrows Y$ as a set-valued mapping, and introduce the unconstrained optimization problem where F is the objective

$$(P) \quad \text{minimize } F(x), \text{ subject to } x \in X.$$

The standard Pareto minimality for this problem is stated in the next definition as the efficiency with respect to the partial order $\leq_K$ induced on Y by K on the basis of the equivalence $y_1 \leq_K y_2$ iff $y_2 - y_1 \in K$.

Definiţia 1 *A point* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ *is a Pareto minimum point for* F*, or a Pareto solution for* (P)*, if there exists* $\varepsilon \in (0, \infty]$ *such that*

$$[F(B(\overline{x}, \varepsilon)) - \overline{y}] \cap -K = \{0\}. \tag{13}$$

If $\operatorname{int} K \neq \emptyset$, $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ *is a weak Pareto minimum point for* F*, or a weak Pareto solution for* (P)*, if there exists* $\varepsilon \in (0, \infty]$ *such that*

$$[F(B(\overline{x}, \varepsilon)) - \overline{y}] \cap -\operatorname{int} K = \emptyset.$$

Obviously, in the above definitions, the case $\varepsilon \in (0, \infty)$ corresponds to the local minima, while the case $\varepsilon = +\infty$ describes the global solutions. We mention that the main results of this work apply to both situations.

It is easy to see that $(\overline{x}, \overline{y})$ is a minimum for F (in any of the above senses) iff it is a minimum of the same type for the epigraphical set-valued map $\tilde{F} : X \rightrightarrows Y$, $\tilde{F}(x) = F(x) + K$.

Consider now a sequence (F_n) of set-valued mappings acting between X and Y. We associate the sequence of optimization problems, with respect to the same order $\leq_K$, as

$$(P_n) \quad \text{minimize } F_n(x), \text{ subject to } x \in X.$$

The main problem we discuss is the following one: having a sequence $(x_n, y_n) \in \operatorname{Gr} F_n$ of Pareto minima for (P_n) (for all n) such that $(x_n, y_n) \to (x, y) \in \operatorname{Gr} F$, what can we say about the point (x, y) in relation with problem (P) when (F_n) are, in a sense, approximations of F?

A well known fact is that, in general, (x, y) is not a Pareto minimum, even under nice convergence properties of (F_n) towards F. Basically, we propose ourselves to describe some general situations when the approximation properties of the sequence (F_n) ensure that (x, y) is a critical point for (P).

We present next the first of the main results, and for this consider the set-valued mappings (F_n), F as the objectives of the problems (P_n) and (P) introduced before.

Teorema 4 *Suppose that X, Y are Asplund spaces, and take $(x, y) \in \operatorname{Gr} F$. Consider a sequence $(x_n, y_n) \to (x, y)$ such that $(x_n, y_n) \in \operatorname{Gr} F_n$ is a minimum of radius $\varepsilon_n > 0$ for F_n for all n. Assume that:*
(i) $\operatorname{Gr} F$ is locally closed at (x, y);
(ii) K is (SNC) at 0, or F^{-1} is (PSNC) at (y, x);
(iii) $\liminf \varepsilon_n > 0$;
(iv) there exists a function $\varphi : [0, \infty) \to [0, \infty)$ with $\lim_{t\to 0} \varphi(t) = \varphi(0) = 0$ such that, for all n, and for all small α

$$\tilde{F}(B(x, \alpha)) \subset \tilde{F}_n(B(x, \alpha + \varphi(\alpha))). \tag{14}$$

Then there exists $v^* \in K^+ \setminus \{0\}$ such that

$$0 \in D^* F(x, y)(v^*),$$

that is, (x, y) is a critical point of F.

The second situation we study uses a new notion of openness, which reads as follows: given a multifunction $F : X \rightrightarrows Y$, a cone $K \subset Y$, a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, and two constants $\alpha, \beta > 0$, one says that F is $(\alpha, \beta)-$open with respect to K at $(\overline{x}, \overline{y})$ if there exists $\varepsilon > 0$ such that, for any $\rho \in (0, \varepsilon)$,

$$B(\overline{y}, \alpha\rho) \subset F(B(\overline{x}, \rho)) + K \cap B(0, \beta\rho). \tag{15}$$

First, we formulate a result concerning the stability of openness with respect to a cone at Lipschitz perturbations in the global case.

Teorema 5 *Let K be a closed convex cone, and $F, G : X \rightrightarrows Y$ be two multifunctions such that $\operatorname{Gr} F$ and $\operatorname{Gr} G$ are locally closed. Suppose that $\operatorname{Dom}(F + G)$ is nonempty and let $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ be such that $\alpha > \beta$. If F is $(\alpha, \gamma)-$open with respect to K at every point of its graph, and if G is $\beta-$Aubin at every point of its graph, then $F + G$ is $(\alpha - \beta, \gamma)-$open with respect to K at every point of its graph.*

A similar result, formulated for the local case, holds, which involves the additional property of sum-stability of the pair (F, G) around the reference point.

The main result in this case is given by the next theorem.

Teorema 6 *Suppose that X, Y are Asplund spaces, and take $(x, y) \in \operatorname{Gr} F$. Take $G_n : X \rightrightarrows Y$ and consider a sequence $(x_n, y_n) \to (x, y)$ such that $(x_n, y_n) \in \operatorname{Gr}(F + G_n)$ is a minimum or $F + G_n$ for all n. Suppose that:*
(i) $\operatorname{Gr} F$ is locally closed at (x, y) and for all n, $\operatorname{Gr} G_n$ is locally closed at every point from its graph close to $(x, 0)$;

(ii) K is (SNC) at 0 or F^{-1} is (PSNC) at (y, x);
(iii) for all n, there is $\beta_n > 0$ such that G_n in β_n-Aubin around every point from its graph close to $(x, 0)$, and $\beta_n \to 0$;
(iv) for all n, the pair (F, G_n) is locally sum-stable around $(x, y, 0)$.
Then there exists $v^* \in K^+ \setminus \{0\}$ such that

$$0 \in D^*F(x, y)(v^*),$$

that is, (x, y) is a critical point of F.

We have **published the article (in ISI journal)**:

- S. Bilal, O. Carja, T. Donchev, A. I. Lazu, Nonlocal problem for evolution inclusions with one-sided Perron nonlinearities, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 113 (3), 1917-1933, 2019. DOI: 10.1007/s13398-018-0589-6.

We have **submitted for publication three papers in ISI journals**:

- O. Carja, A. I. Lazu, *Minimum time and minimum energy for linear control systems*, submitted on June 27, 2019.
- M. Durea, R. Strugariu, *On the sensitivity of Pareto efficiency in set-valued optimization problems*, submitted on June 14, 2019.
- S. Bilal, O. Carja, T. Donchev, N. Javaid, A. I. Lazu, *Qualitative properties of differential inclusions in Banach spaces*, submitted on July 21, 2019.

Other **two papers are in preparation**. All the above articles were financially supported by the project, confirmed by the corresponding text from the *Acknowledgement* section.

The research results obtained in the frame of this project were disseminated at the following **international conferences**: ROMFIN&FSDONA, June 10-15, 2019, Turku, Finland (A. I. Lazu, "*Nonlocal m-dissipative evolution inclusions in general Banach spaces* "), EQUADIFF Conference, July 8-12, 2019, Leiden, Netherlands (A. I. Lazu, "*On the equivalence of minimum time and minimum norm control*"), International conference on Continuous Optimization (ICCOPT 2019), August 3-8, 2019, Berlin, Germany (R. Strugariu, "*Stability of the directional regularity*"), The Fifth Conference of the Mathematical Society of the Republic of Moldova dedicated to the 55th anniversary of the foundation of the Vladimir Andrunachievici Institute of Mathematics and Computer Science - IMCS55, September 28 - October 1, 2019, Chisinau, Republic of Moldova (R. Strugariu, "*On directional stability of mappings*"), International Conference on Applied and Pure Mathematics, Octorber 31 – November 3, 2019, Iaşi (A. I. Lazu, "*Minimum time and minimum energy for linear systems*").

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